Moving The Mean Of A Normal Distribution

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Moving the mean of a normal distribution while leaving the variance unchanged is used extensively in valuing derivative assets. We move the mean of the asset return distribution such that the discounted asset price is a martingale under the new probability measure. Note that a martingale is where the expected asset price at time t is asset price at time s where $t > s$.

The Problem

We are pulling a random number from a normal distribution with a mean of 2.50 and a variance of 4.00. We are given that the probability of the random number being between 1.00 and 3.00 under this probability measure is 37.21%. Our task is (1) to mathematically move the mean of the distribution from 2.50 to -0.50 while leaving the variance unchanged and (2) to calculate the probability that the random number will lie in the above range under this new probability measure using the cumulative normal distribution function.

Legend of Symbols

m	=	Distribution mean - old distribution
n	=	Distribution mean - new distribution
v	=	Distribution variance - both distributions
u	=	Upper bound of integration
l	=	Lower bound of integration
$N[z]$	=	Cumulative standard normal distribution function
$P[l \leq z \leq u]$	=	Probability that random number z will be between l and u

The Current Distribution

The probability density function (pdf) of a continuous random variable is a function that describes the relative likelihood for this random variable to occur at a given point. The probability for the random variable to fall within a particular region is given by the integral of this variable's density over the region. The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one. The equation for the probability density for our normal distribution with mean m and variance v is...

$$f(z) = \frac{1}{\sqrt{2\pi v}} \, e^{-\frac{1}{2v}(z-m)^2} \tag{1}$$

The first moment of the distribution is the expected value of the random variable z. The equation for the first moment of our distribution is...

$$\begin{aligned}
\mathbb{E}\Big[z\Big] &= \int\limits_{-\infty}^{\infty} z \, f(z) \, \delta z \\
&= \int\limits_{-\infty}^{\infty} z \, \frac{1}{\sqrt{2\pi v}} \, e^{-\frac{1}{2v}(z-m)^2} \, \delta z \\
&= \int\limits_{-\infty}^{\infty} z \, \frac{1}{\sqrt{2\pi \times 4}} \, e^{-\frac{1}{2\times 4}(z-2.5)^2} \, \delta z \\
&= 2.50
\end{aligned} \tag{2}$$

The second moment of the distribution is the expected value of the square of the random variable z. The equation for the second moment of our distribution is...

$$\mathbb{E}\Big[z^2\Big] = \int\limits_{-\infty}^{\infty} z^2 \, f(z) \, \delta z$$
$$= 10.25 \tag{3}$$

The distribution mean is the expected value of the random variable z, which is the first moment of the distribution equation (2)...

$$mean = \mathbb{E}\Big[z\Big]$$
$$= 2.50 \tag{4}$$

The distribution variance is the expected value of the square of the random variable z minus the expected value of the random variable z squared, which is the second moment of the distribution equation (3) minus the first moment of the distribution equation (2) squared...

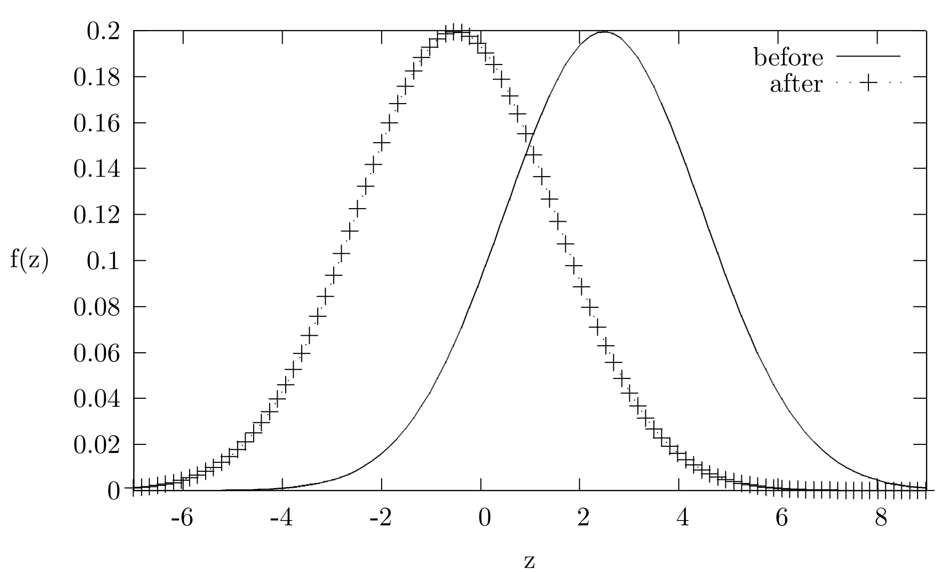
$$variance = \mathbb{E}\Big[z^2\Big] - \Big[\mathbb{E}\Big[z\Big]\Big]^2$$
$$= 10.25 - 2.50^2$$
$$= 4.00 \tag{5}$$

We are given that under the current distribution the probability of the random variate z being between 1.0 and 3.0 where z is normally-distributed with mean 2.50 and variance 4.00 is 37.21%. This statement of probability in equation form is...

$$P\Big[l \le z \le u\Big] = \int\limits_{l}^{u} \frac{1}{\sqrt{2\pi v}} \, e^{-\frac{1}{2v}(z-m)^2} \delta z$$
$$P\Big[1.0 \le z \le 3.0\Big] = \int\limits_{1}^{3} \frac{1}{\sqrt{2\pi \times 4}} \, e^{-\frac{1}{2 \times 4}(z-2.5)^2} \delta z$$
$$= 0.3721 \tag{6}$$

Step One - Mathematically Moving the Mean

Our task is to move the mean of the current distribution from 2.50 to -0.50 while leaving the variance unchanged. The graph below presents the normal distribution before and after the mean is moved from 2.50 to -0.50...



Our task is to derive the function $\xi(z)$ such that when we multiply the probability density under the current distribution by this multiplier the mean of the distribution shifts from 2.50 to -0.50 while keeping the variance unchanged. Our multiplier will be in the following form...

$$\xi(z) = e^{\frac{(n-m)z}{v} - \frac{n^2 - m^2}{2v}} \tag{7}$$

The new probability distribution is the integral of the probability density function for the current distribution, which is equation (1), times the multiplier, which is equation (7). The equation for the new distribution is...

$$\begin{aligned}
new\,distribution &= \int\limits_{-\infty}^{\infty} f(z)\xi(z)\delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(z-m)^2}\, e^{\frac{(n-m)z}{v} - \frac{n^2-m^2}{2v}}\,\delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{z^2}{2v} + \frac{mz}{v} - \frac{m^2}{2v}}\, e^{\frac{nz}{v} - \frac{mz}{v} - \frac{n^2}{2v} + \frac{m^2}{2v}}\,\delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{z^2}{2v} + \frac{nz}{v} - \frac{n^2}{2v}}\,\delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(z-n)^2}\,\delta z
\end{aligned} \tag{8}$$

Note that the probability distribution in equation (8) above has a mean of n and a variance of v.

Step Two - Recalculate the Probability

The new equation (replaces equation (6) above) for the probability that the random variate z is between 1.0 and 3.0 where z is normally-distributed with mean -0.50 (the variable n) and variance 4.00 (the variable v) is...

$$P\Big[l \le z \le u\Big] = \int\limits_{l}^{u} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(z-n)^2}\,\delta z \tag{9}$$

The problem states that we must use the cumulative normal distribution function to solve our problem. To do this we must normalize equation (9) above. Normalizing means that we transform a normal distribution with mean n and variance v to a normal distribution with mean zero and variance one. We normalize our distribution by subtracting the mean and dividing by the standard deviation. The normalizing equation where the variable x is the new random variable is...

$$x = \frac{z-n}{\sqrt{v}} \tag{10}$$

After normalizing the probability density (equation (1)) as a function of the new random variable x becomes...

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \tag{11}$$

The process of normalizing requires us to adjust the bounds of integration in equation (9) above, which after nomalizing becomes...

$$P\Big[l \le x \le u\Big] = \int\limits_{\frac{l-n}{\sqrt{v}}}^{\frac{u-n}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,\delta x \tag{12}$$

The Answer to Our Problem

Equation (12) is the cumulative normal distribution function. To solve our problem we calculate the probability that the random variable x is less than the upper bound of integration and subtract the probability the the random variable is less than the lower bound of integration.

The probability that the random variable is less than 3.00 is...

$$N\left[\frac{u-n}{\sqrt{v}}\right]=N\left[\frac{3.00-(-0.50)}{\sqrt{4.00}}\right]=0.95994 \tag{13}$$

The probability that the random variable is less than 1.00 is...

$$N\left[\frac{l-n}{\sqrt{v}}\right]=N\left[\frac{1.00-(-0.50)}{\sqrt{4.00}}\right]=0.77337 \tag{14}$$

The answer to our problem is equation (13) minus equation (14) which is...

$$P\Bigg[1.00 \leq x \leq 3.00\Bigg]=0.95994-0.77337=0.18657 \tag{15}$$

or...

$$P\Bigg[1.00 \leq x \leq 3.00\Bigg]=18.66\% \tag{16}$$